Mock Exam

21/04/2025, 9:00 am - 12:00 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- You are allowed to use one and only one A4 cheat sheet.

Exercise 1: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{\operatorname{Exp}\left(-\frac{1}{x^2+y^2}\right)\cos(xy)}{x^2 + y^2},$$

where Exp() is the exponential function.

(a) (0.5 points) Compute the gradient $\nabla f(x, y)$.

(b) (1.5 points) Prove that the directional derivatives of f exist at any point and in every direction. Then, compute the directional derivative of f at $(0, 0)$ in the direction of $\mathbf{v} = (v_1, v_2) \neq (0, 0)$.

Exercise 2: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = \begin{bmatrix} x^2 + y^2 - 2 \\ xz - y \\ y^2 - 1 \end{bmatrix}.$$

(a) (0.5 points) Verify whether the Jacobian $JF(1, 1, 1)$ is invertible.

(b) (0.5 points) Use the implicit function theorem to determine if z can be expressed as a function of x and y near $(1, 1, 1)$.

(c) (1 point) Let g be the implicit function, i.e., $z = g(x, y)$. Compute $\mathrm{D}g(1, 1)$.

(d) (1 point) Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be the inverse of F. Compute $\mathrm{D}G(0, 0, 0)$.

Exercise 3: (1 point) Let A be the region in $\mathbb{R}^2$ limited by $\{x = 0\}$, $\{y = 0\}$, and $\{x^2 + y^2 = 1\}$. Compute $\int_A x^2 y \, dx dy$.

Exercise 4: Consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$. Using the parametrization

$$\phi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v), \quad u \in [0, 2\pi], \ v \in [0, \pi],$$

(a) (1 point) Is the given parametrization orientation-preserving when $\mathbb{S}^2$ is oriented by the outward normal?

(b) (1 point) Evaluate the integral of the 2-form $\omega = x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y$ over the sphere $\mathbb{S}^2$.

Exercise 5: Let $\omega = x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y$ be a 2-form in $\mathbb{R}^3$.

(a) (0.5 points) Compute $\mathbf{d}\,\omega$.

(b) (0.5 points) Verify if ω is closed and/or exact.

Exercise 6: (1 point) Let S be the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 1$ oriented by the outward normal. Let $\mathbf{F} = (-y, x, z)$ be a vector field. What is the flux of $\nabla \times \mathbf{F}$ across S?

*The exam on Thursday will contain 2 more bonus questions!